Quaternion-Octonion Unitary Symmetries and Analogous Casimir Operators

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Abstract

An attempt has been made to investigate the global $SU(2)$ and $SU(3)$ unitary flavor symmetries systematically in terms of quaternion and octonion respectively. It is shown that these symmetries are suitably handled with quaternions and octonions in order to obtain their generators, commutation rules and symmetry properties. Accordingly, Casimir operators for $SU(2)$ and $SU(3)$ flavor symmetries are also constructed for the proper testing of these symmetries in terms of quaternions and octonions.

Key Words: $SU(2)$ and $SU(3)$ flavor symmetries, quaternion, octonion and Casimir operators

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1 Introduction

Division algebras, specifically quaternions and octonions, Jordan and related algebras, are described [1] in a conceited manner in unified theories of basic interactions. Quaternions were discovered by Hamilton [2] in 1843 as an illustration of group structure and also applied to mechanics in three-dimensional space. Quaternions have the same properties as complex numbers with the difference that the commutative law is not valid in their case. It gave the importance to quaternions in terms of their possibility to understand the fundamental laws of physics. Rather, the octonions [3, 4] form the widest normed algebra after the algebra of real numbers, complex numbers and quaternions. The octonions are also known as Cayley Graves numbers and also have an algebraic structure defined on the 8-dimensional real vector space in such a way that two octonions can be added, multiplied and divided with the fact that multiplication is neither commutative nor associative. Nevertheless, other expected properties like distributivity and alternativity hold well to the case of octonions. Indeed, Pais [5] pointed out a striking similarity between the algebra of interactions and the split octonion algebra. Furthermore, some attention has been given to octonions [6] in theoretical physics in order to extend the 3+1 space-time to eight dimensional space-time as the consequence to accommodate the ever increasing quantum numbers and internal symmetries related to elementary particles and gauge fields. However, Günaydin and Gürsey [7] has also developed the quark model and color gauge theory in terms of split octonions. Above all, a Casimir operator [8, 9], named after Hendrik Casimir, was described as an important tool in the study [10, 11, 12, 13, 14, 15, 16] of associative and alternative algebras. There is an infinite family of Casimir operators whose members are expressible in terms of a number of primitive Casimirs equal to the rank of the underlying group. Moreover, an algebra of colors suitably handled with octonions has been discussed [17] for two Casimir operators and the two generators of the Cartan sub algebra of the automorphism group $SU(3)$ of the Hilbert space. Further, we have developed [18] the quaternionic formulation of Yang- Mill's field equations and octonionic reformulation of quantum chromo dynamics (QCD) where the resemblance between quaternions and $SU(2)$ and that of octonions and $SU(3)$ gauge symmetry has been discussed. The color group $SU(3)_C$ is embedded with in the octonionic structure of the exceptional groups while the $SU(3)$ flavor group has been discussed in terms of triality property of the octonion algebra. Therefore, we have also derived the relations for different components of isospin with quark states. On the other hand , we [19] have analyzed this formalism to the case of $SU(3)$ flavor group in terms of its splitting to various pairs of $SU(2)$ isospin handled with quaternions. Likewise, the various commutation relations among the generators of $SU(3)$ group and its shift operators have been derived and verified in terms of different isospin multiplets i.e. I, U and V- spins. Keeping in view, the utilities of quaternions and octonions in internal symmetry groups, in this paper, we have made an attempt to investigate the global $SU(2)$ and $SU(3)$ unitary flavor symmetries respectively in terms of quaternions and Octonions. Therefore, we have developed $SU(2)$ flavor symmetry from quaternions to relate isotopic spin and also $SU(3)$ symmetry (the so called eight fold way) from octonions to study the three flavors of quarks and anti-quarks. It is shown that these symmetries are suitably handled with quaternions and octonions in order to obtain their generators, commutation rules and symmetry properties. Consequently, the analogous Casimir operators for $SU(2)$ and $SU(3)$ flour symmetry groups are analyzed and suitably handled respectively with quaternions and octonions. It is also shown that analogous Casimir operators commute with the corresponding generators of $SU(2)$ and $SU(3)$ gauge groups.

2 Quaternion Gauge Theory and Isospin

Let $\phi(x)$ be a quaternionic field (Q field) and expressed as

$$\phi = e_0\phi_0 + e_j\phi_j \ (\forall\, j = 1, 2, 3) \tag{1}$$

where ϕ_0 and ϕ_j are local Hermitian fields while the e_0 and e_j are respectively the real and the imaginary basis of quaternion Q satisfying the following multiplication property

$$e_0^2 = e_0; \quad e_0 e_j = -\delta_{jk}e_0 + \epsilon_{jkl}e_l \ (\forall j, k, l = 1, 2, 3). \tag{2}$$

The quaternion basis elements (e_0, e_1, e_2, e_3) may also be written [20, 21, 22, 23, 24, 25] in terms of 4×4 real or 2×2 complex matrices. The 2×2 correspondence with complex matrices is described [21, 22] as

$$e_1 \Rightarrow \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}; \ e_2 \Rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \ e_3 \Rightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \tag{3}$$

which are expressed as 2×2 Pauli spin Hermitian matrices $\tau_j (\forall j = 1, 2, 3)$ connected to spatial unitary group $SU(2)$ as

$$\tau_1 = -ie_1 \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \ \tau_2 = -ie_2 \Rightarrow \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \ \tau_3 = -ie_3 \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4}$$

However, in quantum Chromodynamics, flavor is a global symmetry where the unitary transformations are independent of space and time. Rather, this symmetry is broken in the electroweak theory and flavor changing processes exist, such as quark decay or neutrino oscillations. $SU(2)$ isospin flavor symmetry at the quark level is denoted by $SU(2)_f$. Therefore, the $SU(2)$ global gauge symmetry may be handled with quaternion gauge formalism in an enthusiastic manner. So, a quaternion spinor ψ transforms as

$$\psi \longmapsto \psi' = U\psi \tag{5}$$

where U is 2×2 unitary matrix and satisfies

$$U^\dagger U = UU^\dagger = UU^{-1} = U^{-1}U = 1. \tag{6}$$

On the other hand, the quaternion conjugate spinor transforms as

$$\overline{\psi} \longmapsto \overline{\psi'} = \overline{\psi}U^{-1}. \tag{7}$$

Therefore, the combination $\psi\overline{\psi} = \overline{\psi}\psi = \psi\overline{\psi'} = \overline{\psi'}\psi$ is an invariant quantity. We may thus write any unitary matrix as

$$U = \exp\left(i\,\hat{H}\right) \ (i = \sqrt{-1}) \tag{8}$$

where $\hat{H}$ is Hermitian $\hat{H}^\dagger = \hat{H}$. Here, the Hermitian 2×2 matrix may be defined in terms of four real numbers, $\alpha_{1,}\alpha_2$, α_3 and α_0 as

$$\hat{H} \quad = \quad \alpha_0 \hat{1} + \tau_j \alpha_j = \alpha_0 e_0 - i\,\alpha_j e_j\,(\forall j = 1,2,3) \tag{9}$$

where $\hat{1}$ is the 2×2 unit matrix and thus we may write the Hermitian matrix $\hat{H}$ as

$$\hat{H} \quad = \quad \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix}. \tag{10}$$

For global gauge transformations both $\alpha_0 = \theta\,(\text{say})$ and $\overrightarrow{\alpha}$ are independent of space-time so that the proton-neutron (or u-d quark) doublet wave function be described [19] as

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix} \tag{11}$$

where $|u>= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|d>= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and equation (11) then transforms as

$$V \begin{pmatrix} u \\ d \end{pmatrix} V^{-1} \equiv U \begin{pmatrix} u \\ d \end{pmatrix} \tag{12}$$

with

$$U = \exp\left(\frac{1}{2}\alpha_j \tau_j\right) \equiv \exp\left(-\frac{1}{2}\alpha_j e_j\right) \equiv \exp(i\,\overrightarrow{\alpha}.\overrightarrow{T}). \tag{13}$$

Here, the isospin $\overrightarrow{T}$ is considered as the analog of the angular momentum and the rotational invariance in internal isotopic spin space implies that the isospin is conserved. The generators of isospin [i.e the $SU(2)$] group are defined as $\hat{T}_j = \frac{1}{2}\tau_j \equiv \frac{1}{2}\,e_j(\forall j = 1,2,3)$ which satisfy the following well known commutation relation

$$\left[\widehat{T}_j, \widehat{T}_k\right] = i\epsilon_{jkl}\widehat{T}_l\,(\forall j,k,l = 1,2,3). \tag{14}$$

In $SU(2)_f$,define the base state $\psi = \begin{pmatrix} u \\ d \end{pmatrix} \equiv \mathbf{2}$, is sometimes called the "fundamental representation". Accordingly, the conjugate state corresponds to the antiparticle (ant-quark or anti-nucleon) states which are described by the complex conjugate (not the Hermitian conjugate) of the $SU(2)_f$ representation since here we require the transformations for column vector ($\frac{\overline{u}}{\overline{d}}$) instead of the row vector (u, d). So, by identifying $\bar{u} \equiv u^\star$ and $\bar{d} \equiv d^\star$, we may write $SU(2)_f$ transformation law for the anti quark doublet as

$$\psi^{\star\prime} \cong \begin{pmatrix} \bar{u}' \\ \bar{d}' \end{pmatrix} \Longleftrightarrow U^\star \psi^\star \cong \exp\left(\alpha_j e_j/2\right) \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}. \tag{15}$$

It should be noted that the quark doublet (u,d) and anti-quark doublet $\left(\bar{u},\bar{d}\right)$ transform differently under $SU(2)_f$ transformations. These two representations are unitary equivalent, so we can consider unitary matrix U_C as

$$U_C \exp\left(\alpha_j e_j/2\right) U_C^{-1} = \exp\left(-\alpha_j e_j/2\right). \tag{16}$$

Here, the unitary matrix U_C satisfies the following conditions i.e.

$$\begin{aligned} U_C\left(e_1\right)U_C^{-1} &= -\,e_1;\\ U_C\left(e_2\right)U_C^{-1} &= e_2;\\ U_C\left(e_3\right)U_C^{-1} &= -\,e_3 \end{aligned} \tag{17}$$

and in order to obtain a convenient unitary representation, one can choose U_C as

$$U_C = -\,e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{18}$$

This implies that the doublet

$$\psi_C = U_C \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} = \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix} \equiv \overline{2}; \tag{19}$$

transforms exactly in the same way as $\psi = \begin{pmatrix} u \\ d \end{pmatrix} \equiv \mathbf{2}$. This result is useful in order to write the familiar table of (Clebsch-Gordan) angular momentum [13] coupling coefficients for combining quark and anti-quark states together. For this, we include the relative minus sign between the $\bar{d}$ and $\bar{u}$ components which has appeared in equation (19). As such,

- **For $\mathbf{T}=\mathbf{0}$**, $\bar{q}q$ combination $\frac{1}{\sqrt{2}}(\bar{u}u+\bar{d}d)\longmapsto \frac{1}{\sqrt{2}}(u^\star u+d^\star d)\cong \psi^\dagger\psi$ where $\frac{1}{\sqrt{2}}$ is used as normalization constant. So, under an $SU(2)_f$ transformation,

$$\psi^\dagger\psi \longmapsto \psi'^\dagger\psi' = \psi^\dagger U^\dagger U\psi = \psi^\dagger\psi; \tag{20}$$

showing that $\psi^\dagger\psi = \frac{1}{\sqrt{2}}(\bar{u}u+\bar{d}d) \equiv |0,0>$ is indeed an $SU(2)_f$ invariant for isospin singlet states $T=0$ and $T_3=0$

- **For $\mathbf{T}=\mathbf{1}$**(isospin triplet), we consider the three quantities φ_j defined as

$$\varphi_j = \psi^\dagger\left(ie_j\right)\psi \ \left(\forall\, j=1,2,3\right) \tag{21}$$

along with an infinitesimal $SU(2)_f$ transformation

$$\psi' = \left(\hat{1}_2 - \alpha_j e_j/2\right)\psi. \tag{22}$$

So, on using equations (11) and (20), we get the iso- triplet for $T=1$ as

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \rightleftharpoons \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{u}d + \bar{d}u) \\ (-i\bar{u}d + i\bar{d}u) \\ (\bar{u}u - \bar{d}d) \end{pmatrix} \tag{23}$$

where $\varphi_3 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d) \equiv |1, 0>$ describes the state corresponding to quantum number isospin ($T = 1; T_3 = 0$) analogous to neutral pion π^0. Similarly, we may assign the isospin quantum numbers to the linear combination of $\varphi_1$and $\varphi_2$as for isospin ($T = 1; T_3 = -1$) analogous to charged pions π^-as

$$\frac{1}{\sqrt{2}}\left(\varphi_1 + i\varphi_2\right) \rightleftharpoons \bar{u}d \equiv |1, -1> \tag{24}$$

and for isospin ($T = 1; T_3 = +1$) analogous to charged pions π^+ as

$$\frac{1}{\sqrt{2}}\left(\varphi_1 - i\varphi_2\right) \rightleftharpoons \bar{d}u \equiv |1, +1> . \tag{25}$$

3 Quaternions and Dirac spinors

Let us define the free particle quaternion Dirac equation [26] for a particle of mass m as

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0. \tag{26}$$

Here, we used the following quaternion valued Weyl representation of γ matrices as it is convenient for ultra relativistic problems i.e

$$\gamma^0 = \begin{bmatrix} 0 & e_0 \\ e_0 & 0 \end{bmatrix}, \equiv \begin{bmatrix} 0 & \hat{1}_2 \\ \hat{1}_2 & 0 \end{bmatrix}, \quad \gamma^j = \begin{bmatrix} 0 & -ie_j \\ ie_j & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & \tau^j \\ -\tau^j & 0 \end{bmatrix} (\forall j = 1, 2, 3). \tag{27}$$

In Weyl representation, we have $\left(\gamma^0\right)^\dagger = \gamma^0$, $\left(\gamma^j\right)^\dagger = -\gamma^j$ and $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0\,(\forall \mu, \nu = 0, 1, 2, 3)$. Rather, at sufficiently low energies [27] the effect of weak interactions dies away and the dominant contribution appears due to the presence of electromagnetic and the strong interactions both of which are Parity conserving interactions. So, we use the simplest representation that closes under Parity $\hat{\mathbb{P}}$ for Dirac spinor field ψ in equation (26) as,

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \cong (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \tag{28}$$

where $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are the simplest representations of the Lorentz group $SL(2, C)$ so that an object transforming in the $(\frac{1}{2}, 0)$ representation is called a left-chiral ψ_L Weyl spinor. Similarly, an object transforming in the $(0, \frac{1}{2})$ representation of Lorentz group $SL(2, C)$, is called a right-chiral Weyl spinor ψ_R. It is to be noted that the meaning assigned to the technical word 'chiral' (or so called chirality means handedness)) is associated with the matrix γ^5 which anti-commutes with all Dirac matrices (27) i.e.

$$\left\{\gamma^5\,,\,\gamma^\mu\right\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0\ (\forall \mu = 0, 1, 2, 3). \tag{29}$$

So, from the anti-commutation relation $\{\gamma^{\mu},\, \gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 0\,(\forall\mu,\nu = 0,1,2,3)$ between the γmatrices (27), it can be easily seen that the matrix

$$\gamma^{5} = i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} = \begin{pmatrix} -e_{0} & 0 \\ 0 & e_{0} \end{pmatrix} \equiv \begin{pmatrix} -\hat{1}_{2} & 0 \\ 0 & \hat{1}_{2} \end{pmatrix} \tag{30}$$

satisfies Eq. (29) along with the following properties

$$\left(\gamma^{5}\right)^{\dagger} = \gamma^{5}; \quad \left(\gamma^{5}\right)^{2} = \hat{1}_{2} \equiv 1. \tag{31}$$

As such, we may define the "left-handed" and "right-handed" projection operators (which are generally used as the terms "left-chiral" and "right-chiral") as

$$\widehat{P}_{L} = \frac{1}{2}\left(1-\gamma^{5}\right); \quad \widehat{P}_{R} = \frac{1}{2}\left(1+\gamma^{5}\right); \tag{32}$$

which shows that the operators $\widehat{P}_{L/R}$ project onto the left/right-chiral Weyl spinor as

$$\widehat{P}_{L}\psi = \frac{1}{2}\left(1-\gamma^{5}\right)\psi = \begin{pmatrix} \psi_{L} \\ 0 \end{pmatrix} \equiv \psi_{L}; \quad \widehat{P}_{R}\psi = \frac{1}{2}\left(1+\gamma^{5}\right)\psi = \begin{pmatrix} 0 \\ \psi_{R} \end{pmatrix} \equiv \psi_{R}. \tag{33}$$

Thus, we may write the suitable Dirac Lagrangian for left-right chiral spinors as,

$$\mathcal{L}_{D} = \psi_{L}^{\dagger}\overline{e_{\mu}}\partial_{\mu}\psi_{L} + \psi_{R}^{\dagger}\overline{e_{\mu}}\partial_{\mu}\psi_{R} - m\left(\psi_{L}^{\dagger}\psi_{R} + \psi_{R}^{\dagger}\psi_{L}\right); \tag{34}$$

where $\mu = 0,1,2,3$. Here, the Euler-Lagrange equations are obtained on considering ψ_{L} and $\psi_{L}^{\star}$ independent (since they are complex fields) and similarly for ψ_{R}, $\psi_{R}^{\star}$. Then, performing the variations with respect to $\psi_{L}^{\star}$ and $\psi_{R}^{\star}$, we get that the Dirac equation (26) is equivalent to the pair of equations

$$\begin{aligned} \overline{e_{\mu}}\partial_{\mu}\psi_{L} =& m\psi_{R}; \\ \overline{e_{\mu}}\partial_{\mu}\psi_{R} =& m\psi_{L}. \end{aligned} \tag{35}$$

These two pairs of equations decouple for the $m = 0$ (i.e for zero mass) particles like neutrinos. We may also write the Lagrangian in a compact form for which it is convenient to define the Dirac adjoint spinor as $\overline{\psi} = \psi^{\dagger}\gamma^{0} \equiv \left(\psi_{R}^{\dagger}, \psi_{L}^{\dagger}\right)$ in chiral representation. hence, the compact and consistent form of the Dirac Lagrangian may be written as,

$$L_{D} = \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi. \tag{36}$$

4 Flavor $SU(3)_{f}$ and Octonions

Gell-Mann and Ne'eman [28] were the first to propose $SU(3)_{f}$ as the correct generalization of isospin $SU(2)_{f}$ to include third quantum number the strangeness along with the isospin. The Gell-Mann λ

matrices are used for the representations of the infinitesimal generators of the special unitary group called $SU(3)$. This group consists eight linearly independent generators $F_A = \frac{\lambda_A}{2} (\forall\, A = 1, 2, 3, ...8)$ which satisfy the following commutation relation as

$$[F_A\,, F_B] = i f_{ABC} F_C \tag{37}$$

where f_{ABC} is the structure constants of $SU(3)$ like ϵ_{jkl} of $SU(2)$ and is completely antisymmetric. The Gell-Mann matrices $\lambda_A (\forall\, A = 1, 2, 3, ...8)$ are defined as

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}; \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix}
\end{aligned} \tag{38}$$

which satisfy the following properties

$$\begin{aligned}
(\lambda_A)^\dagger &= \lambda_A; \\
Tr(\lambda_A) = 0 \quad Tr(\lambda_A \lambda_B) &= 2\delta_{AB}; \\
[\lambda_A, \lambda_B] &= 2 i f^{ABC} \lambda_C; \\
\{\lambda_A, \lambda_B\} &= \frac{4}{3}\delta_{AB} + 2\, d_{ABC} \lambda_C.
\end{aligned} \tag{39}$$

Here d_{ABC} is totally symmetric tensor and it is required only to obtain one of the Casimir operators of $SU(3)$ symmetry group. Gellmann λ matrices obviously act on three component column vectors as the generalization of the two component isospinors of $SU(2)$. The generators of $SU(3)_f$ connect three quarks namely the up (u), down (d). and strange (s) quarks so that we may consider unitary 3×3 transformations among them as

$$\psi' = W\psi; \tag{40}$$

where ψ now stands for the three component column vector

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix};$$

and W is the 3×3 unitary matrix of determinant1. The representation provided by this triplet of states is called the fundamental representation of $SU(3)_f$. An infinitesimal $SU(3)$ matrix may then be defined as

$$W_{infl} = \hat{1}_3 + i\chi; \tag{41}$$

where $\hat{1}_3$is the unit matrix of order 3 and χ is 3×3 matrix and satisfies the properties of octonions [17, 19]. The relation between Gell - Mann λ matrices and octonion units are given [18] as

$$\begin{aligned} e_1 &\Rightarrow i\lambda_1,\ e_2 \Rightarrow i\lambda_2,\ e_3 \Rightarrow i\lambda_3 \longmapsto e_A \Longleftrightarrow i\lambda_A; \quad (\forall\, A = 1,2,3); \\ e_4 &\Rightarrow \frac{i}{2}\lambda_4,\ \ e_5 \Rightarrow \frac{i}{2}\lambda_5, \Longleftrightarrow e_A = \frac{i}{2}\lambda_A; \quad (\forall\, A = 4,5,); \\ e_6 &\Rightarrow -\frac{i}{2}\lambda_6,\ \ e_7 \Rightarrow -\frac{i}{2}\lambda_7, \Longleftrightarrow e_A = -\frac{i}{2}\lambda_A; \quad (\forall\, A = 6,7,); \\ e_0 &\Longleftrightarrow \frac{\sqrt{3}}{2}\lambda_8; \end{aligned} \tag{42}$$

where $e_A(A = 1,2,.....,7)$ are imaginary octonion units and e_0 is the real octonion basis element corresponding to unity. Here, a set of octets (e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7) are known as the octonion [6] basis elements and satisfy the following multiplication rules

$$\begin{aligned} e_0 = 1;\ e_0e_A = e_Ae_0 = e_A; \\ e_Ae_B = -\delta_{AB}e_0 + f_{ABC}e_C.\ (\forall\, A,\ B,\ C = 1,\ 2, ...,\ 7). \end{aligned} \tag{43}$$

The structure constants f_{ABC} is completely antisymmetric and takes the value 1 for the following combinations,

$$\begin{aligned} f_{ABC} &= +1; \\ \forall\ \ (ABC) &= (123), (471), (257), (165), (624), (543), (736). \end{aligned} \tag{44}$$

Thus, it is worth noting to suitably handle the octonions in order to reexamine the $SU(3)_f$ symmetry group and its properties.

5 Generators and Casimir Invariants

A very useful concept in group representation theory is that of Casimir operators [16, 29] although there is no general agreement [15] on a unique definition of Casimir operator[8, 9]. Nonetheless, there exists a set of invariant matrices [15] which can be constructed from the contractions of the generators of a Lie group. Such invariant matrices are called the Casimir operators [15, 29] and hence commute with all the generators of a Lie group. A familiar example, from ordinary quantum mechanics, is the Casimir operator $\hat{L}^2$ for the rotation group. It commutes with the three generators $\overrightarrow{L}$ of the rotation group and its eigenvalues $l(l+1)\hbar^2$ label the 'simplest' states of particles with angular momentum. Hence, a Casimir operator is a nonlinear function of the generators that commutes with all of the generators of the group. The adjoint representation of the algebra is given by the structure constants , which are always real i.e.

$$
\begin{aligned}
\left[\widehat{X}_A\,,\widehat{X}_B\right] &= i f_{ABC}\,\widehat{X}_C; \\
\left[\widehat{X}_A\,,\left[\widehat{X}_B\,,\widehat{X}_C\right]\right] &= i f_{BCD}\left[\widehat{X}_A\,,\widehat{X}_D\right] = -f_{BCD}f_{ADE}\widehat{X}_E.
\end{aligned} \tag{45}
$$

If there are N generators, then we get a $N\times N$ matrix representation in the adjoint representation. From the Jacobi identity,

$$
\left[\widehat{X}_A\,,\left[\widehat{X}_B\,,\widehat{X}_C\right]\right] + \left[\widehat{X}_B\,,\left[\widehat{X}_C\,,\widehat{X}_A\right]\right] + \left[\widehat{X}_C\,,\left[\widehat{X}_A\,,\widehat{X}_B\right]\right] = 0; \tag{46}
$$

we get the similar relation for the structure constants i.e. $f_{BCD}f_{ADE} + f_{ABD}f_{CDE} + f_{CAD}f_{BDE} = 0$. The states of the adjoint representation correspond to the generators $|\widehat{X}_a >$. We may now consider the simple Lie-algebras whose generators satisfy the commutation relation

$$
\left[\mathbb{T}^a,\mathbb{T}^b\right] = \sum_k f^{abc}\mathbb{T}^c; \tag{47}
$$

where we use Hermitian generators $\mathbb{T}^a$ in order to choose the Killing form [16] proportional to δ^{ab}i.e.

$$
Trace\,\mathbb{T}^a\mathbb{T}^b \propto \delta^{ab}; \tag{48}
$$

with a positive and representation dependent proportionality constant that can be fixed from the choice of the group. With this convention the structure constants f^{abc} are real and completely anti-symmetric and the generators of the adjoint representation are related [16] to the structure constants

$$
(\mathbb{T}_A)^a_{bc} = -\,i\,f^{abc}. \tag{49}
$$

From the commutation relations (45), one can write the anti-symmetric structure constant as

$$
f^{abc} = -\,\frac{i}{\hat{C}}\;Trace\left[\left[\mathbb{T}^a,\mathbb{T}^b\right]\mathbb{T}^c\right]; \tag{50}
$$

where the generators $\mathbb{T}^a$ act on $N\times N$ matrices of a semi simple Lie (Special Unitary $SU(N)$) group of N^2-1 generators under the condition $Trace\,\mathbb{T}^a = 0$ and $\hat{C}$ is Casimir operator. So, the adjoint representation of the group plays an important role to examine the Casimir operators due to the equation (47). Hence, the commutation relation (46) does not really diminish the number of generators of the adjoint representation. Rather, it is a different way of writing the Jacobi identity [16]. Casimir operator for corresponding group, every expression $\hat{C}$ in the $\mathbb{T}^a$'s commutes with all the basis elements of the algebra i.e. $\left[\hat{C},\mathbb{T}^a\right] = 0$. In general,$\hat{C}$ does not belong to the algebra as it is not linear in $\mathbb{T}^a$. The number of generators required to give a complete set of Casimir invariants is equal to the rank of the group. Casimir operators are used to label irreducible representations of the Lie algebra. The utility of Casimir operators arises from the fact that all states in a given representation assume the same value for a Casimir operator. This is because the states in a given representation are connected by the action of the generators of the Lie algebra and such generators commute with the Casimir operators. This property may then be used to label representations in terms of the values of the Casimir operators.

6 Casimir Operator for Quaternion $SU(2)$ group

In Section-2, we have already stated the correspondence between the quaternion units e_j and generators of $SU(2)$ isotopic spin group. One of the important property of $SU(2)$group is the existence of invariant operator (namely the total angular momentum, $\hat{\mathbb{J}}^2$ in internal isotopic spin space) which commutes with all of the generators of $SU(2)$ group. For compact groups, the Killing form is just the Kronecker delta. So, for $SU(2)$ group, the Casimir invariant is then simply the sum of the square of the generators $\widehat{\mathbb{J}}_x$,$\widehat{\mathbb{J}}_y$;$\widehat{\mathbb{J}}_z$ of the algebra. i.e., the Casimir invariant is given by

$$\hat{\mathbb{J}}^2 = \widehat{\mathbb{J}}_x^2 + \widehat{\mathbb{J}}_y^2 + \widehat{\mathbb{J}}_z^2. \tag{51}$$

The Casimir eigenvalue in a irreducible representation is $j^2 = j(j+1)$ where for brevity we use the natural units $c = \hbar = 1$. Since the number of Casimir operators corresponds to the rank of the group, there is only one Casimir operator in $SU(2)$ which obviously commutes with all the generators. The Casimir operator is proportional to the identity element (i.e. the scalar quaternion unit e_0). This constant of proportionality can be used to classify the representations of the Lie group and is also related to the mass or (iso)spin. The proportional constant for (iso) spin $SU(2)$ group denotes the "square" of total (iso)spin $I^2 = I(I+1)$ for $I = 1/2$. We have already stated that the generators of $SU(2)$ isospin [19] in terms of quaternions are $\hat{T}_j = \frac{1}{2}\tau_j \equiv \frac{1}{2}\, e_j (\forall j = 1, 2, 3)$ and satisfy the commutation relation (14) analogous to Eq. (47). So, the Casimir operator for $SU(2)$ quaternion group is

$$\hat{\mathbb{I}}^2 = \widehat{\mathbb{I}}_x^2 + \widehat{\mathbb{I}}_y^2 + \widehat{\mathbb{I}}_z^2 \tag{52}$$

with eigenvalues $I(I+1)$. It is isomorphic to the For $SO(3)$ group for which the Casimir Operator is the total angular momentum $\hat{\mathbb{J}}^2$ given by equation (51). So, the members of the irreducible representations of isospin (or angular momentum) are represented by the 2 numbers I and $\mathfrak{m}$ associated respectively r with the total (iso) spin and its $z-$component i.e.

$$\begin{aligned} \hat{\mathbb{I}}^2 |I, \mathfrak{m}> &= I(I+1)\, |I, \mathfrak{m}> \\ \widehat{\mathbb{I}_z} |I, \mathfrak{m}> &= \mathfrak{m}\, |I, \mathfrak{m}> . \end{aligned} \tag{53}$$

The individual states are connected by the “ladder operators”

$$\begin{aligned} \widehat{\mathbb{I}}_\pm &= \widehat{\mathbb{I}_x} \pm \widehat{\mathbb{I}_y}; \\ \widehat{\mathbb{I}}_\pm |I, \mathfrak{m}> &= \sqrt{(I \mp \mathfrak{m})(I \pm \mathfrak{m} + 1)}\, |I, \mathfrak{m} \pm 1>; \end{aligned} \tag{54}$$

where I can be determined by the formula $n = 2I + 1$ (n is the dimension of the generator) and for a given I, the value of $\mathfrak{m}$ can be $-I, -I+1, ..., I-1, I$; so that it has $2I+1$ degenerate states for a given I. For $SU(2)$; $n = 2$, $I = 1/2$, and $\mathfrak{m} = -\frac{1}{2}$ or $\mathfrak{m}$=$+\frac{1}{2}$ (an iso doublet). For continuous symmetries, the resulting quantum numbers (eigenvalues of the “total” Casimir operator – the total isospin) are obtained from the appropriate “addition” of the quantum numbers of the individual representations being added. The rank of $SU(2)$ group is $n - 1 = 2 - 1 = 1$ leading to the definition of Casimir Operator that there exists only one non linear invariant operator (i.e. the Casimir Operator) for quaternion $SU(2)$ group. Hence, the Casimir Operator for $SU(2)$ group may then be constructed from the square of the isospin

(52) as

$$\widehat{\mathsf{C}}=\sum_{a=1}^{3}I_{a}^{2}=-\frac{1}{4}\sum_{a=1}^{3}e_{a}^{2}=-\frac{1}{4}(e_{1}^{2}+e_{2}^{2}+e_{3}^{2})\cong\frac{3}{4}\,e_{0}\equiv\frac{3}{4}; \tag{55}$$

where $e_1^2 = e_2^2 = e_3^2 = -1$; $e_0^2 = e_0 = 1$. is the unique Casimir operator. This invariant operator obviously commutes with all the generators of quaternion $SU(2)$ group i.e.

$$\left[\widehat{\mathsf{C}}\,,\widehat{\mathbb{I}_{x}}\right]=\left[\widehat{\mathsf{C}}\,,\widehat{\mathbb{I}_{y}}\right]=\left[\widehat{\mathsf{C}}\,,\widehat{\mathbb{I}_{z}}\right]=\left[\widehat{\mathsf{C}}\,,\widehat{\mathbb{I}_{\pm}}\right]=0. \tag{56}$$

7 Split Octonions

Let us start with an octonion $\mathcal{O}$, which is expanded [30] in a basis (e_0, e_A) as

$$\mathcal{O}=\mathcal{O}^{0}+\sum_{A=1}^{A=7}O^{A}e_{A}(\forall A,B.C,=1,2,3,....,7); \tag{57}$$

where $e_A(A = 1, 2, ..., 7)$: the imaginary octonion units and $(e_0 = 1)$: the real octonion unit satisfy the properties given by Eqs. (43) and (44) for non-commutative and non-associative octonion algebra. So, we write the octonion conjugation $(\bar{\mathcal{O}})$, octonion norm $|\mathcal{O}|$ and octonion inverse $(\mathcal{O}^{-1})$ [6, 17, 18, 19, 20] as

$$\overline{\mathcal{O}}=\mathcal{O}^{0}-\sum_{A=1}^{A=7}O^{A}e_{A}(\forall A,B.C,=1,2,3,....,7); \tag{58}$$

$$|\mathcal{O}|=\mathbb{N}\left(\mathcal{O}\right)=<\mathcal{O}|\mathcal{O}>=\overline{\mathcal{O}}\mathcal{O}=\mathcal{O}\overline{\mathcal{O}}=(\mathcal{O}^{0})^{2}+\sum_{A=1}^{A=7}(\mathcal{O}^{A})^{2}; \tag{59}$$

$$\mathcal{O}^{-1}=\frac{\overline{\mathcal{O}}}{|\mathcal{O}|}=\frac{\overline{\mathcal{O}}}{\mathbb{N}\left(\mathcal{O}\right)}; \tag{60}$$

$$\mathcal{O}^{-1}\mathcal{O}=\mathcal{O}\mathcal{O}^{-1}=1. \tag{61}$$

For three octonions $x, y, z \in \mathcal{O}$, the non-vanishing associator is defined by

$$<x,\,y,\,z>=x\,(y\,z)-(x\,y)\,z. \tag{62}$$

It vanishes only for those non-commutating combinations for which the structure constant $f^{ABC} = 1$ i.e.

$$\begin{aligned}<e_{A},e_{B},\,e_{C}>&=(e_{1},e_{2},e_{3})=(e_{4},e_{7},e_{1})=(e_{2},e_{5},e_{7})=(e_{1},e_{6},e_{5})\\&=(e_{6},e_{2},e_{4})=(e_{5},e_{4},e_{3})=(e_{7},e_{3},e_{6})=0.\end{aligned} \tag{63}$$

It justifies that an octonion resembles to $SU(3)$ symmetry consisting seven non Abelian $SU(2)$ symmetry groups analogous to quaternions. However, the octonion algebra over the field of complex numbers is visualized as the Split Octonion algebra [1, 17] with its split base units defined as

$$u_0 = \frac{1}{2}(e_0 + ie_7)\,; \quad u_0^\star = \frac{1}{2}(e_0 - ie_7)\,;$$
$$u_m = \frac{1}{2}(e_m + ie_{m+3})\,; \quad u_m^\star = \frac{1}{2}(e_m + ie_{m+3})\,(\forall m = 1,2,3); \tag{64}$$

where $(\star)$ denotes the complex conjugation and $(i = \sqrt{-1})$ (the imaginary unit) commutes with all $e_A\,(\forall\,A = 1,2,3,....,7)$. In equation (64) u_0, $u_0^\star$, u_j, $u_j^\star$ are defined as the bi-valued representations of quaternion units e_0 , e_1, e_2, e_3 satisfy $e_je_k = -\delta_{jk} + \epsilon_{jkl}e_l$ $(\forall j,k,l = 1,2,3)$. Thus, The split octonion basis elements (64) satisfy the following multiplication rules

$$\begin{aligned}
u_iu_j = -u_ju_i = \epsilon_{ijk}u_k^\star, &\quad u_iu_j = -u_j^\star u_i^\star = \epsilon_{ijk}u_k;\\
u_iu_j^\star = -\delta_{ij}u_0, &\quad u_i^\star u_j = -\delta_{ij}u_0^\star;\\
u_0u_i = u_iu_0^\star = u_i, &\quad u_0^\star u_i^\star = u_i^\star u_0 = u_i^\star;\\
u_iu_0 = u_0u_i^\star = 0, &\quad u_i^\star u_0^\star = u_0^\star u_i = 0;\\
u_0u_0^\star = u_0u_0^\star = 0 &\quad u_0^2 = u_0,\ u_0^{\star 2} = u_0^\star.
\end{aligned} \tag{65}$$

These relations (65) are invariant [17] under G_2 group of automorphism of octonions. Under the $SU(3)$ subgroup of automorphism group G_2 that leaves the imaginary unit e_7(or equivalently the idempotents $u_0$and $u_0^\star$) invariant, and the $u_m$and $u_m^\star(\forall m = 1,2,3)$ transform like [17] a triplet (**3**) and anti-triplet ($\mathbf{3}^\star$) of $SU(3)_f$. Unlike octonions, the split octonion algebra contains zero divisors and is therefore not a division algebra.

8 Split Octonions and $SU(3)_f$ Symmetry

The split octonion algebra may now be regarded as the Lie algebra of $SU(3)$. So, we may write the complex Octonion ψ in terms of split octonion basis as,

$$\psi = u^\dagger\phi + u_0^\star\phi_0; \tag{66}$$

where

$$u^\dagger = (u_1^\star u_2^\star u_3^\star)\,, \ \ \phi = \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}. \tag{67}$$

Here, we may illustrate the transformations of the associated split octonion u with Gell Mann λ matrices as,

$$
\begin{aligned}
u^{\dagger}\lambda_1 u =& u_1^{\star}u_2+u_2^{\star}u_1;\\
u^{\dagger}\lambda_2 u =& -i\left(u_1^{\star}u_2-u_2^{\star}u_1\right);\\
u^{\dagger}\lambda_3 u =& u_1^{\star}u_1-u_2^{\star}u_2=-u_1^{\star}\lambda_8 u=-u^{\dagger}Yu;\\
u^{\dagger}\lambda_4 u =& u_3^{\star}u_1+u_1^{\star}u_3;\\
u^{\dagger}\lambda_5 u =& -i\left(u_3^{\star}u_1-u_1^{\star}u_3\right);\\
u^{\dagger}\lambda_6 u =& u_2^{\star}u_3+u_3^{\star}u_2;\\
u^{\dagger}\lambda_7 u =& -i\left(u_2^{\star}u_3-u_3^{\star}u_2\right);\\
u^{\dagger}\lambda_8 u =& u_1^{\star}u_1+u_2^{\star}u_2-2u_3^{\star}u_3.
\end{aligned}
\tag{68}
$$

and

$$
\begin{aligned}
\frac{1}{\sqrt{3}}u^{\dagger}\lambda_3 u =& \frac{1}{\sqrt{3}}\left(u_1^{\star}u_1-u_2^{\star}u_2\right)=-u_1^{\star}\lambda_8 u=-u^{\dagger}Yu;\\
u^{\dagger}\frac{1}{2}\left(\lambda_3+\frac{1}{\sqrt{3}}\lambda_8\right)u =& u^{\dagger}Qu=2\left(u_1^{\star}u_1-u_3u_3^{\star}\right).
\end{aligned}
\tag{69}
$$

As such, we may establish the step up (shift) operators as

$$
\begin{aligned}
u^{\dagger}\left(\lambda_1+i\lambda_2\right)u =& u^{\dagger}I_+u=u_1^{\star}u_2;\\
u^{\dagger}\left(\lambda_4+i\lambda_5\right)u =& u^{\dagger}U_+u=u_3^{\star}u_1;\\
u^{\dagger}\left(\lambda_6+i\lambda_7\right)u =& u^{\dagger}V_+u=u_2^{\star}u_3.
\end{aligned}
\tag{70}
$$

and we may construct the two such independent Casimir (operators) for the split octonion $SU(3)$ group.Similarly the step down (shift) operators may be expressed as

$$
\begin{aligned}
u^{\dagger}\left(\lambda_1-i\lambda_2\right)u =& u^{\dagger}I_-u=u_2^{\star}u_1;\\
u^{\dagger}\left(\lambda_4-i\lambda_5\right)u =& u^{\dagger}U_-u=u_1^{\star}u_3;\\
u^{\dagger}\left(\lambda_6-i\lambda_7\right)u =& u^{\dagger}V_-u=u_3^{\star}u_2.
\end{aligned}
\tag{71}
$$

The split octonion group thus decomposes $SU(3)$ into singlets $(u_0\,;\,u_0^{\star})$, a triplet u_i and an anti- triplet $u_i^{\star}$. Consequently, the $SU(3)$ flavor symmetries are suitably handled with split octonions and may be explored with $I-$, $U-$ and $V-$ spins (flavors) of $SU(2)$ group as

$$I_1=u_1^{\star}u_2+u_2^{\star}u_1;\quad I_2=-i\left(u_1^{\star}u_2-u_2^{\star}u_1\right);\quad I_3=u_1^{\star}u_1-u_2^{\star}u_2\,(I-\,Spin); \tag{72}$$

$$V_1=u_3^{\star}u_1+u_1^{\star}u_3;\quad V_2=-i\left(u_3^{\star}u_1-u_1^{\star}u_3\right);\quad V_3=-i(u_2^{\star}u_2-u_3^{\star}u_3)\,(V-\,Spin); \tag{73}$$

$$U_1=u_2^{\star}u_3+u_3^{\star}u_2;\quad U_2=-i\left(u_2^{\star}u_3-u_3^{\star}u_2\right);\quad U_3=-i((u_1^{\star}u_1-2u_3^{\star}u_3))\,(U-\,Spin); \tag{74}$$

along with the spin octonion valued hyper charge is described as

$$Y=\frac{1}{\sqrt{3}}\lambda_8=\frac{1}{\sqrt{3}}\left(u_1^{\star}u_1+u_2^{\star}u_2-2u_3^{\star}u_3\right). \tag{75}$$

Likewise, the spin octonion valued shift operators for $I-$, $U-$ and $V-$ spins (flavors) of $SU(2)$ group are also analyzed as

$$\begin{aligned}I_{+} = & u_{1}^{\star}u_{2}; \quad I_{-}=u_{2}^{\star}u_{1};\\ V_{+} = & u_{3}^{\star}u_{1}; \quad V_{-}=u_{1}^{\star}u_{3}\\ U_{+} = & u_{2}^{\star}u_{3}; \quad U_{-}=u_{2}^{\star}u_{3}.\end{aligned} \tag{76}$$

The commutation relations between I_+ and I_-, U_+ and U_- and V_+ and V_- are then be described

$$\begin{aligned}\left[I_{+},\,I_{-}\right]=2\,I_{3}; & \quad \left[I_{3},\,I_{\pm}\right]=\pm I_{\pm};\\ \left[U_{+},\,U_{-}\right]=2\,U_{3}; & \quad \left[U_{3},\,U_{\pm}\right]=\pm U_{\pm};\\ \left[V_{+},\,V_{-}\right]=2\,V_{3} & \quad \left[V_{3},\,V_{\pm}\right]=\pm V_{\pm};\\ \left[I_{+},\,V_{+}\right]=\left[I_{+},U_{+}\right]=\left[U_{+},V_{+}\right]= & 0;\\ \left[Y,\,I_{3}\right]=\left[Y,\,U_{3}\right]=\left[Y,\,V_{3}\right]= & 0.\end{aligned} \tag{77}$$

Accordingly, the Charge Q is described as

$$Q = 2\left(u_{1}^{\star}u_{1}-u_{3}u_{3}^{\star}\right). \tag{78}$$

These equations satisfy all the commutation relations of I, U and V - spin multiplets of $SU(3)$ flavor group along with the Gell- Mann Nishimija relation $Q=\frac{Y}{2}+I_{3}$.

9 Casimir Operators for split Octonion $SU(3)$ group

The group $SU(3)$ deals the global $SU(3)$ of quark flavors and the local $SU(3)_C$gauge symmetry of quantum Chromodynamics (QCD). Here we are interested in the former case where the fundamental representation is the triplet of (lowest mass) quarks flavors u, d,s. In the latter case the fundamental representation is the triplet, $A=1,2,3$ (or red, green, blue), with one such triplet for each of the6 quark flavors. Let us start with suitably handled split octonion $SU(3)$ symmetry associated with triplet of (lowest mass) quarks flavors u, d,s. So, formulas (53) and (54) can be generalized to the case of $SU(3)$ symmetry where we have $n=3$, $I=1$, and $\mathfrak{m}=+1,0,-1$ forming an iso-triplet of triplet of quarks flavors u, d,s. The group $SU(3)$ is of rank two (i.e. $n-1=3-1=2$). So, it is to be noted that the individual members of an irreducible representation of $SU(3)$ are labeled by 2 constants. These are defined by the third component of Iso-spin $\widehat{\mathbb{I}}_{3}=\frac{1}{2}\lambda_{3}$ and hypercharge $\widehat{Y}=\frac{1}{2\sqrt{3}}\lambda_{8}$both are diagonal. Consequently, we may construct [13] two independent the Casimir invariant tensors (operators) for split Octonion $SU(3)$ group as

$$\widehat{\boldsymbol{C}}_{2}=-\frac{2}{3}\,i\,f_{ABC}\,\widehat{F_{A}}\,\widehat{F_{B}}\,\widehat{F_{C}}=\sum_{A=1}^{A=8}\widehat{F}_{A}^{2} \tag{79}$$

and

$$\widehat{\boldsymbol{C}}_{3} = d_{ABC}\widehat{F_{A}}\,\widehat{F_{B}}\,\widehat{F_{C}} \tag{80}$$

where $F_{A}\left(\forall A,B,C=1,2,3.....,8\right)$ are suitably handled with split octonions in terms of $I-$, $U-$, and $V-$ spins $SU(3)$ flavor symmetry. It is easily verified that $\widehat{\boldsymbol{C}}_{2}$ and $\widehat{\boldsymbol{C}}_{3}$ commute with all the generators

$\widehat{F}_A$ and the shift operators $I-$, $U-$, and $V-$ spins of split octonion $SU(3)$ group as well. These Casimir operators $\widehat{\boldsymbol{C}}_p(\forall p = 2, 3)$ also commute with the Hamiltonian $\hat{H}$ of strong interactions.

10 Conclusion

The foregoing analysis provides a fundamental representation of quaternions and the split octonion algebra in order to investigate the flavor $SU(2)$ and $SU(3)$ symmetries. Instead of Pauli Spin matrices for $SU(2)$ group we have used the compact notations of quaternions while for the case of $SU(3)$ symmetry we have established the connection between the Gellmann λ matrices and octonion basis elements. The isospin symmetry and the quark-anti quark symmetries are suitably handled with quaternions and octonions. It is shown that the structure constants of $SU(3)$ symmetry resembles with those of multiplication identities of octonion basis elements. This result is useful for the use of familiar table of (Clebsch-Gordan) angular momentum coupling coefficients for combining quark and anti-quark states together by using quaternions and octonions. It is remarkable that the $SU(3)$ group of split octonion is an invariant sub group of octonion automorphism group G_2. So, it is necessary to handle whole $SU(3)$ symmetry in terms of split octonions in an enthusiastic manner. Since Casimir operators for split octonion gives explicit basis for various sub-algebras, a set of commuting operators in terms of various permutations of octonions and isospin multiplets is the useful for the unique and consistent representation for $SU(2)$ and $SU(3)$ flavor symmetries and the theory of strong interactions as well. Consequently, the analogous Casimir operators for $SU(2)$ and $SU(3)$ flour symmetry groups are analyzed and suitably handled respectively with quaternions and octonions. It is also shown that analogous Casimir operators commute with the corresponding generators of $SU(2)$ and $SU(3)$ gauge groups.

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References

[1] F. Gürsey and Chia-Hsiung Tze, **"On the role of Division Jordan and related algebras in Particle Physics"**, World Scientific, Singapore, (1996).

[2] W. R. Hamilton, **"Elements of quaternions"**, Chelsea Publications Co., NY (1969).

[3] A. Cayley, **"On Jacobi's elliptic functions, in reply to the Rev. B. Bronwin; and on quaternions"**, Phil. Mag. **26** (1845), 208.

[4] R. P. Graves, **"Life of Sir William Rowan Hamilton"**, 3 volumes, Arno Press, New York (1975).

[5] A. Pais, **"Remark on the Algebra of Interactions"**, Phys. Rev. Lett. **7** (1961), 291.

[6] J. C. Baez, **"The Octonions"**, Bull. Amer. Math. Soc. **39** (2001), 145.

[7] M. Günaydin and F. Gürsey, **"Quark structure and octonions"**, J. Math. Phys., **14** (1973), 1651.

[8] H. Casimir, **"Liber die Konstruction einer zu den irreduzibelen Darstellungen halbeinfacher Kontinuierlicher Gruppen gehorigen Differenzialgleichung"**, Proc. Nederl. Akad. Wetensch, Amstd. **34** (1931), 844.

[9] H. Casimir and B. L. van der Waerden, **"Algebraischer Beweis der Vollständigen Reduzibilität der Darstellungen halbeinfacher Liescher Gruppen"**, Math. Ann., **111** (1935), 1.

[10] A. M. Perelomov and V. S. Popov, " **Casimir Operators for Semisimple Lie Groups**", Math. USSR - Izvestija, **2** (1968), 1313.

[11] J. J. Sullivan and T. H. Siddal, "**Casimir Operators, Duality and the Point Groups**", J. Math. Phys., **33** (1992), 1964.

[12] A. J. Mountain, "**Invariant Tensors and Casimir Operators for Simple Compact Lie Groups**", J. Math. Phys., **39** (1998), 5601.

[13] E. Zeidler, **"Quantum Field Theory III: Gauge Theory"**, Springer (2010).

[14] S. Okubo, "**Casimir invariants and vector operators in simple and classical Lie algebras**," J. Math. Phys. **18** (1977), 2382.

[15] P. Cvitanovic, **"Group Theory: Birdtracks, Lie's and Exceptional Groups**", Princeton (2008).

[16] T. van Ritbergen, A. N. Schellenkens and J. A. M. Vermaseren, "**Group Theory Factors for Feynmann Diagrams**", Int. J. Mod. Phys., **A14** (1999), 41.

[17] M. Günaydin, **"Octonionic Hilbert spaces, the Poincaré group and SU(3)"**, J. Math. Phys. **17** (1976) 1875.

[18] Pushpa, P. S. Bisht, Tianjun Li and O. P. S. Negi, **"Quaternion Octonion Reformulation of Quantum Chromodynamics"**, Int. J. Theor. Phys., **50** (2011), 594.

[19] Pushpa, P. S. Bisht, Tianjun Li and O. P. S. Negi, **"Quaternion-Octonion SU(3) Flavor Symmetry"**, eprint- arXiv:1107.1559 [physics:gen-ph]; Int. J. Theor. Phys., DOI: 10.1007/s10773-011-1062-x.

[20] S.L. Altmann, "**Rotations, quaternions and double groups**", Clarendon Press, Oxford (1986).

[21] C. Lanczos, "**The variational principles of mechanics**", University of Toronto Press (1970).

[22] J. Kronsbein, "**Kinematics, Quaternions, Spinors and Pauli's Spin Matrices**"; Amer. J. Phys., **35** (1967), 335.

[23] R. Anderson and Girish C. Joshi; "**Quaternions and the heuristic role of mathematical structures in physics**"; Phys.Essays, **6** (1993), 308.

[24] D. Widdos, "Quaternionic Algebra described by Sp(1) representations", Quart. J. Math., **54** (2003), 463.

[25] I Abonyi, J. F. Bito and J. K. Tar, "**A quaternion representation of the Lorentz group for classical physical applications** ", J. Phys. A Math. Gen., **24** (1991), 3245.

[26] Seema Rawat and O. P. S. Negi, **"Quaternion Dirac Equation and Supersymmetry"**, Int. J. Theor. Phys., **48** (2009) 2222.

[27] M. Maggiore, **"A Modern Introduction to Quantum Field Theory"**, Oxford University press, (2005).

[28] Murray Gell-Mann and Y. Ne'eman, "**The Eight fold Way**", W. A. Benjamin, (1964).

[29] T. Ohlsson, **"Relativistic Quantum Physics: From Advanced Quantum Mechanics to introductry Quantum Field Theory"**, Cambridge University Press (2011).

[30] P. S. Bisht, Shalini Dangwal and O. P. S. Negi, **"Unified Split Octonion Formulation of Dyons"**, Int. J. Theor. Phys., **47** (2008) 2297.